

# MULTIPLICATIVE CONGRUENCES WITH VARIABLES FROM SHORT INTERVALS

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ABSTRACT. Recently, several bounds have been obtained on the number of solutions to congruences of the type

$$(x_1 + s) \dots (x_\nu + s) \equiv (y_1 + s) \dots (y_\nu + s) \not\equiv 0 \pmod{p}$$

modulo a prime  $p$  with variables from some short intervals. Here, for almost all  $p$  and all  $s$  and also for a fixed  $p$  and almost all  $s$ , we derive stronger bounds. We also use similar ideas to show that for almost all primes, one can always find an element of a large order in any rather short interval.

## 1. INTRODUCTION

For a prime  $p$ , let  $\mathbb{F}_p$  be the field of residues modulo  $p$ . Also, denote  $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ . For integers  $h \geq 3$  and  $\nu \geq 1$  and elements  $s \in \mathbb{F}_p$  and  $\lambda \in \mathbb{F}_p^*$ , we denote by  $K_\nu(p, h, s)$  the number of solutions of the congruence

$$(1) \quad \begin{aligned} (x_1 + s) \dots (x_\nu + s) &\equiv (y_1 + s) \dots (y_\nu + s) \not\equiv 0 \pmod{p}, \\ 1 &\leq x_1, \dots, x_\nu, y_1, \dots, y_\nu \leq h. \end{aligned}$$

Recently, a series of bounds on  $K_\nu(p, h, s)$  as well as on the number of solutions of a one-sided congruence

$$(2) \quad \begin{aligned} (x_1 + s) \dots (x_\nu + s) &\equiv \lambda \pmod{p}, \\ 1 &\leq x_1, \dots, x_\nu \leq h. \end{aligned}$$

have been obtained, see [3, 4, 8] and references therein.

In particular, it is shown in [4] that for any fixed integer  $\nu \geq 3$  we have

$$K_\nu(p, h, s) \leq \left( \frac{h^\nu}{p^{\nu/e_\nu}} + 1 \right) h^\nu \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right),$$

where

$$e_\nu = \max\{\nu^2 - 2\nu - 2, \nu^2 - 3\nu + 4\}.$$

Here we use and develop further some ideas of [3, 4] and obtain stronger bounds on  $K_\nu(p, h, s)$

- for almost all  $p$  and all  $s$ ;
- for a fixed  $p$  and almost all  $s$ .

For this purpose, we also consider the following equation with complex  $\sigma \in \mathbb{C}$ :

$$(3) \quad \begin{aligned} (x_1 + \sigma) \dots (x_\nu + \sigma) &= (y_1 + \sigma) \dots (y_\nu + \sigma) \neq 0, \\ 1 \leq x_1, \dots, x_\nu, y_1, \dots, y_\nu &\leq h. \end{aligned}$$

which is an analogue of the congruence (1).

We denote by  $K_\nu(h, \sigma)$  the number of solutions of (3). Here we give an asymptotic formula for  $K_\nu(h, \sigma)$  that holds for almost all rational  $\sigma$  and all irrational  $\sigma$ , which could be of independent interest.

Finally, in Section 5 we give applications of our bounds of  $K_\nu(p, h, s)$ , and underlying ideas, to the existence of elements of large order in short intervals.

We recall that the notations  $A \ll B$ ,  $B \gg A$  and  $A = O(B)$  are both equivalent to the statement that the inequality  $|A| \leq cB$  holds with some constant  $c > 0$ . Throughout the paper, any implied constants in the symbols ' $\ll$ ', ' $\gg$ ' and ' $O$ ' may depend on the integer parameter  $\nu \geq 1$  and sometimes on some other explicitly mentioned parameters and are absolute otherwise.

As usual, we use  $\pi(T)$  to denote the number of primes  $p \leq T$ .

## 2. PRELIMINARIES

**2.1. Background on geometry of numbers.** Recall that a lattice in  $\mathbb{R}^n$  is an additive subgroup of  $\mathbb{R}^n$  generated by  $n$  linearly independent vectors. Take an arbitrary convex compact and symmetric with respect to 0 body  $D \subseteq \mathbb{R}^n$ . Recall that, for a lattice  $\Gamma \subseteq \mathbb{R}^n$  and  $i = 1, \dots, n$ , the  $i$ th successive minimum  $\lambda_i(D, \Gamma)$  of the set  $D$  with respect to the lattice  $\Gamma$  is defined as the minimal number  $\lambda$  such that the set  $\lambda D$  contains  $i$  linearly independent vectors of the lattice  $\Gamma$ . Obviously,  $\lambda_1(D, \Gamma) \leq \dots \leq \lambda_n(D, \Gamma)$ . We need the following result given in [2, Proposition 2.1] (see also [18, Exercise 3.5.6] for a simplified form that is still enough for our purposes).

**Lemma 1.** *We have,*

$$\#(D \cap \Gamma) \leq \prod_{i=1}^n \left( \frac{2i}{\lambda_i(D, \Gamma)} + 1 \right).$$

**2.2. Resultant bound.** Let  $\text{Res}(P_1, P_2)$  denote the resultant of two polynomials  $P_1$  and  $P_2$ .

**Lemma 2.** *Let  $H \geq 1$ ,  $\rho, \vartheta \in \mathbb{R}$ , and let  $M, N \geq 2$  be fixed integers. Assume also that one of the following conditions hold:*

- (i)  $\rho \geq 0$ ;
- (ii)  $\vartheta \geq 0$ ;
- (iii)  $\rho + \vartheta \geq -1$ .

*Let  $P_1(Z)$  and  $P_2(Z)$  be non-constant polynomials with integer coefficients*

$$P_1(Z) = \sum_{i=0}^{M-1} a_i Z^{M-1-i} \quad \text{and} \quad P_2(Z) = \sum_{i=0}^{N-1} b_i Z^{N-1-i}$$

*such that*

$$\begin{aligned} |a_i| &< H^{i+\rho}, \quad i = 0, \dots, M-1, \\ |b_i| &< H^{i+\vartheta}, \quad i = 0, \dots, N-1. \end{aligned}$$

*Then*

$$\text{Res}(P_1, P_2) \ll H^{(M-1+\rho)(N-1+\vartheta)-\rho\vartheta},$$

*where the implicit constant in  $\ll$  depends only on  $M$  and  $N$ .*

**2.3. Background on algebraic integers.** Let  $\mathbb{K}$  be a finite extension of  $\mathbb{Q}$  and let  $\mathbb{Z}_{\mathbb{K}}$  be the ring of integers in  $\mathbb{K}$ . We recall that the logarithmic height of an algebraic number  $\alpha$  is defined as the logarithmic height  $H(P)$  of its minimal polynomial  $P$ , that is, the maximum logarithm of the largest (by absolute value) coefficient of  $P$ .

We need a bound of Chang [5, Proposition 2.5] on the divisor function in algebraic number fields.

**Lemma 3.** *Let  $\mathbb{K}$  be a finite extension of  $\mathbb{Q}$  of degree  $d = [\mathbb{K} : \mathbb{Q}]$ . For any nonzero algebraic integer  $\gamma \in \mathbb{Z}_{\mathbb{K}}$  of logarithmic height at most  $H \geq 2$ , the number of pairs  $(\gamma_1, \gamma_2)$  of algebraic integers  $\gamma_1, \gamma_2 \in \mathbb{Z}_{\mathbb{K}}$  of logarithmic height at most  $H$  with  $\gamma = \gamma_1 \gamma_2$  is at most  $\exp(O(H/\log H))$ , where the implied constant depends on  $d$ .*

Very often we use Lemma 3 for  $d = 1$  when it gives the classical bound on the usual divisor function.

Now recall that the Mahler measure of a nonzero polynomial

$$P(Z) = a_d Z^d + \dots + a_1 Z + a_0 = a_d \prod_{j=1}^d (Z - \xi_j) \in \mathbb{C}[Z]$$

is defined as

$$M(P) = |a_d| \prod_{j=1}^d \max\{1, |\xi_j|\},$$

see [14, Chapter 3, Section 3]

We recall the following estimates, that follows immediately from a much more general [14, Theorem 4.4]:

**Lemma 4.** *For any nonzero polynomial  $P$  of degree  $d$  the following inequality holds*

$$2^{-d}e^{H(P)} \leq M(P) \leq (d+1)^{1/2}e^{H(P)}.$$

**Corollary 5.** *For any nonzero polynomials  $Q_1, Q_2 \in \mathbb{C}[Z]$  we have*

$$H(Q_1Q_2) = H(Q_1) + H(Q_2) + O(d),$$

where  $d = \deg(Q_1Q_2)$ .

We also use the following result from [4].

**Lemma 6.** *For any positive integer  $\nu$  there is a constant  $C_\nu$  such that the following holds. If  $P_1, P_2 \in \mathbb{Z}[Z]$ ,  $P = P_1P_2$ ,*

$$P(Z) = \sum_{j=0}^{\nu} u_j Z^{\nu-j}$$

*and for some  $A > 0$  and  $h > 0$  the coefficients of the polynomial  $P$  satisfy the inequalities*

$$u_0 \neq 0, \quad |u_j| \leq Ah^j, \quad j = 0, \dots, \nu,$$

*then the polynomial  $P_1$  has the form*

$$P_1(Z) = \sum_{j=0}^{\mu} v_j Z^{\mu-j}$$

*with*

$$v_0 \neq 0, \quad |v_j| \leq C_\nu Ah^j \quad (j = 0, \dots, \mu).$$

**2.4. Sieve arguments.** We start with recalling the following result of Iwaniec [12]:

**Lemma 7.** *Let  $q_1, \dots, q_r$  be  $r \geq 2$  distinct primes. Then the number of consecutive integers each divisible by at least one of  $q_1, \dots, q_r$  is  $O(r^2 \log^2 r)$ .*

**Corollary 8.** *There is an absolute constant  $c > 0$  such that for any positive integer  $m \geq 2$  and  $a \in \mathbb{N}$  there are at least  $c\sqrt{m}/\log m$  multiplicatively independent numbers  $x \in [a, a+m)$ .*

*Proof.* Without loss of generality we can assume that  $a > 1$ . By Lemma 7, one can take  $r \gg \sqrt{m}/\log m$  so that for any primes  $q_1, \dots, q_r$  there is a number  $x \in [a, a+m)$  not divisible by  $q_1, \dots, q_r$ . We show the existence of  $r+1$  multiplicatively independent numbers  $x \in [a, a+m)$  using recursive procedure. First, we take an arbitrary integer  $x_1 \in$

$[a, a + m)$  and an arbitrary prime divisor  $q_1$  of  $x_1$ . Assuming that for some  $i = 1, \dots, r$  the numbers  $x_1, \dots, x_i$  and prime divisors  $q_j$  of  $x_j$  are chosen we use Lemma 7 to take  $x_{i+1} \in [a, a + m)$  not divisible by  $q_1, \dots, q_i$  and an arbitrary prime divisor  $q_{i+1}$  of  $x_{i+1}$ . Let integers  $n_1, \dots, n_{r+1}$  satisfy the equality

$$x_1^{n_1} \dots x_{r+1}^{n_{r+1}} = 1.$$

Since  $x_1$  is the only number from  $x_1, \dots, x_{r+1}$  that is divisible by  $q_1$  we deduce that  $n_1 = 0$ . Similarly, as  $x_2$  is the only number from  $x_2, \dots, x_{r+1}$  that is divisible by  $q_2$  we conclude that  $n_2 = 0$ , and so on. Finally, we get  $n_1 = \dots = n_{r+1} = 0$  as required.  $\square$

We also recall the celebrated Brun–Titchmarsh theorem, see [13, Theorem 6.6].

**Lemma 9.** *For any integer  $q$  and real  $y$  with  $y \geq 2q > 0$  the number  $\pi(x, q, 1)$  of primes  $p \leq y$  with  $p \equiv 1 \pmod{q}$  does not exceed*

$$\pi(x, q, 1) \leq \frac{2y}{\varphi(q) \log(y/q)} \left( 1 + O\left( \frac{1}{\log(y/q)} \right) \right).$$

**2.5. Distribution of divisors of shifted primes.** We need several results about divisors of shifted primes which follows from the arguments of Edős and Murty [9].

First of all we note that by [9, Theorem 2] we have:

**Lemma 10.** *For any function  $\eta(z) > 0$  with  $\eta(z) \rightarrow 0$  as  $z \rightarrow \infty$  and for  $T \rightarrow \infty$ , for all but  $o(\pi(T))$  primes  $p \leq T$  the number  $p - 1$  has no divisor in  $[T^{1/2-\eta(T)}, T^{1/2+\eta(T)}]$ .*

Furthermore, it is easy to see that the arguments used in the proof of [9, Lemma 1] also lead to the following result (in particular, using the notation of [9], one can notice that the fact that  $\nu = -\log \varepsilon(x)$  is not used in the proof).

**Lemma 11.** *There is an absolute constant  $c_0$  such that for any  $0 < \alpha \leq \gamma \leq 1$  for all but  $c_0(\alpha\gamma^{-1} + 1/\log T)\pi(T)$  primes  $p \leq T$  the product of all prime factors of  $p - 1$  that are smaller than  $T^\alpha$  is at most  $T^\gamma$ .*

**Corollary 12.** *For any constant  $c$  and function  $\eta(z) > 0$  with  $\eta(z) \rightarrow 0$  as  $z \rightarrow \infty$  and for  $T \rightarrow \infty$ , for all but  $o(\pi(T))$  primes  $p \leq T$  the product of all prime factors of  $p - 1$  that are smaller than  $T^{c\eta(T)/\log(1/\eta(T))}$  is at most  $T^{\eta(T)/2}$ .*

We remark that [16, Theorem 2] gives a stronger version of Lemma 11 in some range of parameters  $\alpha$  and  $\gamma$ . However for our applications, it is essential to have an unrestricted range of parameters  $\alpha$  and  $\gamma$ , thus Lemma 11 is more suitable for our goal.

**Lemma 13.** *Let  $0 < \alpha \leq \beta < 1 \leq \lambda$ . Then for  $T \rightarrow \infty$ , for all but at most  $(1/\lambda + o(1))\pi(T)$  primes  $p \leq T$  the number of prime divisors  $q$  of  $p - 1$  satisfying  $q \geq T^\alpha$  does not exceed*

$$\#\{q \text{ prime} : q \mid p - 1, q \geq T^\alpha\} \leq \frac{2\lambda \log(\beta/\alpha)}{1 - \beta} + \frac{1}{\beta}.$$

*Proof.* Denote by  $N(p)$  the number of primes  $q$  dividing  $p - 1$  with  $T^\alpha \leq q < T^\beta$ . We have

$$\sum_{p \leq T} N(p) = \sum_{T^\alpha \leq q < T^\beta} \#\{p \leq T : p \equiv 1 \pmod{q}\}.$$

By Lemma 9, for  $q < T^\beta$  we have

$$\pi(T, q, 1) \leq (1 + o(1)) \frac{2T}{(1 - \beta)q \log T}$$

as  $T \rightarrow \infty$ . Hence,

$$\sum_{p \leq T} N(p) \leq (1 + o(1)) \frac{2T}{(1 - \beta) \log T} \sum_{T^\alpha \leq q < T^\beta} \frac{1}{q}.$$

Next, by the Mertens formula, see [13, Equation (2.15)],

$$\sum_{T^\alpha \leq q < T^\beta} \frac{1}{q} = \log(\beta/\alpha) + o(1)$$

Therefore,

$$\sum_{p \leq T} N(p) \leq \frac{2T}{(1 - \beta) \log T} (\log(\beta/\alpha) + o(1)).$$

Denoting

$$\mathcal{P} = \left\{ p \leq T : N(p) > \frac{2\lambda \log(\beta/\alpha)}{1 - \beta} \right\},$$

we get  $\#\mathcal{P} \leq (1/\lambda + o(1))T/\log T$ . Observing that for a prime  $p \leq T$  the number of prime divisors  $q \geq p^\beta$  is at most  $1/\beta$ , we complete the proof.  $\square$

**Lemma 14.** *Let  $\alpha > 0$ . Then for  $T > 0$ , for all but  $O(T^{1-\alpha})$  primes  $p \leq T$  the number  $p - 1$  has no divisors of the form  $q^2$  with an integer  $q \geq T^\alpha$ .*

*Proof.* The number of primes  $p \leq T$  satisfying  $p \equiv 1 \pmod{q^2}$  does not exceed  $T/q^2$ . Using the inequality

$$\sum_{q \geq T^\alpha} \frac{T}{q^2} \leq 2T^{1-\alpha},$$

the result follows.  $\square$

**2.6. Additive relations in multiplicative groups.** Let  $\Gamma \subseteq \mathbb{C}^*$  be a multiplicative group of rank  $r$  and let  $a_1, \dots, a_n \in \mathbb{Z}$ . We consider the equation

$$(4) \quad a_1x_1 + \dots + a_nx_n = 1, \quad x_1, \dots, x_n \in \Gamma.$$

Recall that a solution of (4) is *non-degenerate* provided no subsum equals zero:

$$\sum_{j \in \mathcal{I}} a_j x_j \neq 0 \quad \text{for } \mathcal{I} \subseteq \{1, \dots, n\}.$$

We use the following result of Evertse-Schlickewei-Schmidt [10].

**Lemma 15.** *The number of non-degenerate solutions of (4) is at most  $\exp(c(n)r)$ .*

Recall that a solution of (4) is non-degenerate provided no subsum equals zero:

$$\sum_{j \in \mathcal{I}} a_j x_j \neq 0 \quad \text{for } \mathcal{I} \subseteq \{1, \dots, n\}.$$

**Corollary 16.** *Let  $\Gamma \subseteq \mathbb{C}^*$  be as in Lemma 15 and let  $\mathcal{A} \subseteq \mathbb{C}$  be a finite set of cardinality  $\#\mathcal{A} = m$ . For  $a_1, \dots, a_n \in \mathbb{Z}^*$  the number of solutions of the equation*

$$(5) \quad a_1x_1 + \dots + a_nx_n = 0, \quad x_1, \dots, x_n \in \Gamma \cap \mathcal{A}.$$

*is at most  $O(m^{\lfloor n/2 \rfloor} + \exp(O(r))m^{\lfloor (n-1)/2 \rfloor})$ , where the implied constants may depend on  $n$ .*

*Proof.* In what follows, the implied constants may depend on  $n$ .

We prove the statement by induction on  $n$ . For  $n = 1, 2$  the statement is trivial. We assume that for some  $k \geq 3$  the statement holds for all  $n < k$ , and we now prove it for  $n = k$ . We can fix  $x_n = b \in \Gamma \cap \mathcal{A}$  such that the number  $J$  of solutions of (5) is not greater than  $m$  times the number of solutions of the equation

$$(6) \quad a_1y_1 + \dots + a_{n-1}y_{n-1} = 1, \quad y_1, \dots, y_{n-1} \in \Gamma_0 \cap \mathcal{A}_0,$$

where  $\Gamma_0 = \langle G \cup \{-a_nb\} \rangle \subseteq \mathbb{C}^*$  (the group generated by  $G \cup \{-a_nb\}$ ) and

$$\mathcal{A}_0 = \{x(-a_nb)^{-1} : x \in \mathcal{A}\}.$$

To each solution of (6) we attach a subset (possibly empty)  $I \subseteq \{1, \dots, n-1\}$  with the largest cardinality such that

$$\sum_{j \in I} a_j y_j = 0.$$

If for a given solution there are several such subsets, then for this solution we attain one of these subsets.

Given a subset  $\mathcal{I} \subseteq \{1, \dots, n-1\}$  (including an empty subset) we collect together those solutions of (6) for which  $\mathcal{I}$  has been attained. There exists a fixed  $\mathcal{I} \subseteq \{1, \dots, n-1\}$  such that

$$(7) \quad J \ll mJ_0,$$

where  $J_0$  is the number of solutions of (6) corresponding to the set  $\mathcal{I}$ . We have

$$(8) \quad \sum_{j \in I} a_j y_j = 0, \quad y_j \in \Gamma_0 \cap \mathcal{A}_0$$

and

$$(9) \quad \sum_{j \in \{1, \dots, n-1\} \setminus I} a_j y_j = 1, \quad y_j \in \Gamma_0.$$

In particular,  $\mathcal{I}$  is a proper subset of  $\{1, \dots, n-1\}$ . Observe that by the maximality of  $\mathcal{I}$  the solutions considered in (9) are non-degenerate. If  $\#\mathcal{I} = n-2$ , then (9) has at most one solution, so that the quantity  $J_0$  is bounded by the number of solutions of (8). Hence, by the induction hypothesis and by (7) we have

$$\begin{aligned} J &\ll m(m^{\lfloor (n-2)/2 \rfloor} + \exp(O(r))m^{\lfloor (n-3)/2 \rfloor}) \\ &\ll m^{\lfloor n/2 \rfloor} + \exp(O(r))m^{\lfloor (n-1)/2 \rfloor}, \end{aligned}$$

and the result follows.

Let now  $\#\mathcal{I} \leq n-3$ . By Corollary 16, the number of non-degenerate solutions of (9) is bounded by  $\exp(O(r))$ . Furthermore, since  $\#\mathcal{I} \leq n-3$ , by the induction hypothesis the number of solutions of (8) is  $O(m^{\lfloor (n-3)/2 \rfloor} + \exp(O(r))m^{\lfloor (n-4)/2 \rfloor})$ . Therefore, by (7) we have

$$\begin{aligned} J &< \exp(c(n)r)m^{\lfloor (n-1)/2 \rfloor} + \exp(c(n)r)m^{\lfloor (n-2)/2 \rfloor} \\ &= \frac{\exp(c(n)r)}{m}m^{\lfloor n/2 \rfloor} + \exp(c(n)r)m^{\lfloor (n-1)/2 \rfloor}, \end{aligned}$$

for some constant  $c(n)$  that depends only on  $n$ .

If  $\exp(c(n)r) \geq m$ , then the trivial estimate  $J \leq m^n$  implies

$$J \leq \exp(nc(n)r)$$

and are done in this case. If  $\exp(c(n)r) < m$ , then

$$J < m^{\lfloor n/2 \rfloor} + \exp(c(n)r)m^{\lfloor (n-1)/2 \rfloor}$$

and the result follows.  $\square$

**Lemma 17.** *Let  $\beta \in \mathbb{C}$  and  $m \in \mathbb{Z}_+$ . Consider a set*

$$\mathcal{A} \subseteq \{\beta + j : 1 \leq j \leq m\} \subseteq \mathbb{C}$$



with  $\#\mathcal{A} > m^\tau$ . Then there is a multiplicatively independent subset  $\mathcal{A}_0 \subseteq \mathcal{A}$  of size

$$\#\mathcal{A}_0 > c(\tau) \log m,$$

where  $c(\tau) > 0$  depends only on  $\tau$ .

*Proof.* Denote  $\Gamma = \langle \mathcal{A} \rangle \subseteq \mathbb{C}^*$  the multiplicative group generated by  $\mathcal{A}$ . If  $\mathcal{A}_0 \subseteq \mathcal{A}$  is a maximal multiplicatively independent subset, clearly for each  $x \in \mathcal{A}$  we have  $x^k \in \langle \mathcal{A}_0 \rangle$  for some positive integer  $k$ . Hence

$$\text{rank } \Gamma = r \leq \#\mathcal{A}_0.$$

We note that for an integer  $n \geq 1$  the sums  $z = x_1 + \dots + x_n$ ,  $x_1, \dots, x_n \in \mathcal{A}$  take values in a set  $\mathcal{Z}_n$  of cardinality  $\#\mathcal{Z}_n \leq nm$ . Let  $N_n(z)$  be the number of such representations. Then,

$$\begin{aligned} & \#\{x_1 + \dots + x_n - x_{n+1} - \dots - x_{2n} = 0, x_1, \dots, x_{2n} \in \mathcal{A}\} \\ &= \sum_{z \in \mathcal{Z}_n} N_n(z)^2 \geq \frac{1}{\#\mathcal{Z}_n} \left( \sum_{z \in \mathcal{Z}_n} N_n(z) \right)^2 = \frac{1}{\#\mathcal{Z}_n} (\#\mathcal{A})^{2n} \geq \frac{(\#\mathcal{A})^{2n}}{nm}. \end{aligned}$$

Applying Corollary 16, we get that

$$\frac{(\#\mathcal{A})^{2n}}{m} < \exp(C(n)r) (\#\mathcal{A})^n.$$

for some constant  $C(n)$  that depends only on  $n$ . Since  $\#\mathcal{A} > m^\tau$ , we derive

$$m^{\tau n - 1} < \exp(C(n)r).$$

Taking  $n = \lceil 2\tau^{-1} \rceil$ , it follows that  $r > c(\tau) \log m$  for some positive constant  $c(\tau)$  that depends only on  $\tau$ .  $\square$

### 3. MULTIPLICATIVE AND POLYNOMIAL CONGRUENCES AND EQUATIONS

**3.1. More general congruences.** To estimate  $K_\nu(p, h, s)$  we sometimes have to study a more general congruence. For a prime  $p$ , an integer  $\nu \geq 1$ , and vectors

$$\begin{aligned} \mathbf{h} &= (h_1, \dots, h_{2\nu}) \in \mathbb{N}^{2\nu}, \\ \mathbf{s} &= (s_1, \dots, s_{2\nu}) \in \mathbb{F}_p^{2\nu}, \\ \mathbf{e} &= (e_1, \dots, e_{2\nu}) \in \{-1, 1\}^{2\nu}, \end{aligned}$$

we denote by  $K_\nu(p, \mathbf{e}, \mathbf{h}, \mathbf{s})$  the number of solutions of the congruence

$$\begin{aligned} (x_1 + s_1)^{e_1} \dots (x_{2\nu} + s_{2\nu})^{e_{2\nu}} &\equiv 1 \pmod{p}, \\ 1 \leq x_j &\leq h_j, \quad j = 1, \dots, 2\nu. \end{aligned}$$

The following result from [4] relates  $K_\nu(p, \mathbf{e}, \mathbf{h}, \mathbf{s})$  and  $K_\nu(p, h, s_j)$ ,  $j = 1, \dots, \nu$ . (The proof is almost instant if one expresses  $K_\nu(p, h, \mathbf{s})$  via multiplicative character sums and uses the Hölder inequality.)

**Lemma 18.** *We have*

$$K_\nu(p, \mathbf{e}, \mathbf{h}, \mathbf{s}) \leq \prod_{j=1}^{2\nu} K_\nu(p, h_j, s_j)^{1/2\nu}.$$

**3.2. Multiplicative congruences and polynomial congruences with small coefficients.** First we derive a certain condition on  $s$  for which the congruence (1) has many solutions with

$$(10) \quad \{x_1, \dots, x_\nu\} \cap \{y_1, \dots, y_\nu\} = \emptyset,$$

In fact this has already appeared as a part of the argument in [4], however here we present it in a self-contained form and in a much large generality that we need here:

**Lemma 19.** *Let  $h < 0.5p^{1/\nu}$ . Assume that for some integer  $s$  there are at least  $N$  solutions to (1) that satisfy the condition (10). Then there are  $Nh^{-1} \exp(O(\log h / \log \log h))$  distinct non-constant polynomials*

$$R(Z) = A_1 Z^{\nu-1} + \dots + A_{\nu-1} Z + A_\nu \in \mathbb{Z}[Z],$$

with

$$|A_i| \leq \binom{\nu}{i} h^i, \quad i = 1, \dots, \nu,$$

and such that  $R(s) \equiv 0 \pmod{p}$ .

*Proof.* We associate with any solution

$$\mathbf{x} = (x_1, \dots, x_\nu) \quad \text{and} \quad \mathbf{y} = (y_1, \dots, y_\nu)$$

of (1) that also satisfy (10), the polynomials

$$P_{\mathbf{x}}(Z) = (x_1 + Z) \dots (x_\nu + Z),$$

and then set

$$(11) \quad R_{\mathbf{x}, \mathbf{y}}(Z) = P_{\mathbf{x}}(Z) - P_{\mathbf{y}}(Z).$$

In particular, since  $R_{\mathbf{x}, \mathbf{y}}(s) \equiv 0 \pmod{p}$  and  $h < 0.5p^{1/\nu}$  it follows that  $R_{\mathbf{x}, \mathbf{y}}(Z)$  is not a constant polynomial (for otherwise it is identical to zero which is impossible by (10)). Clearly each polynomial  $R_{\mathbf{x}, \mathbf{y}}(Z)$  satisfies all the required condition.

By the Dirichlet pigeon-hole principle we have at least  $N/h$  solutions with the same  $x_1 = x_1^*$ . We claim that any polynomial  $R$  induced by these solutions occurs at most  $\exp(c_0(\nu) \log h / \log \log h)$  times for some

$c_0(\nu)$  depending only on  $\nu$ . Indeed, fix  $R$  and assume that  $R = R_{\mathbf{x}, \mathbf{y}}$ . Let  $M = R(-x_1^*)$  and  $z_i = -x_1^* + y_i$ ,  $i = 1, \dots, \nu$ . We have

$$(12) \quad M = -P_{\mathbf{x}}(-x_1^*) - P_{\mathbf{y}}(-x_1^*) = -P_{\mathbf{y}}(-x_1^*) = -z_1 \dots z_{\nu}.$$

Using the well-known bound on the divisor function (a special case of Lemma 3 below), we see that the number of solutions to (12) is bounded by  $\exp(c_0(\nu) \log h / \log \log h)$ . Each solution determines the numbers  $y_1, \dots, y_{\nu}$  and the polynomial  $P_{\mathbf{x}}$ , and for each  $P$  there are at most  $(\nu - 1)!$  solutions of (1) with  $P = P_{\mathbf{x}}$ . This concludes the proof.  $\square$

**3.3. Common solutions to many congruences.** We recall the following result from [4]:

**Lemma 20.** *Let  $\gamma \in (0, 1)$  and let  $I$  and  $J$  be two intervals containing  $h$  and  $H$  consecutive integers, respectively, and such that*

$$h \leq H < \frac{\gamma p}{15}.$$

*Assume that for some integer  $s$  the congruence*

$$y \equiv sx \pmod{p}$$

*has at least  $\gamma h + 1$  solutions in  $x \in I$ ,  $y \in J$ . Then there exist integers  $a$  and  $b$  with*

$$|a| \leq \frac{H}{\gamma h}, \quad 0 < b \leq \frac{1}{\gamma},$$

*such that*

$$s \equiv a/b \pmod{p}.$$

We also need the following estimate:

**Lemma 21.** *There is an absolute constant  $C_0 > 0$  such that if for some  $s$  there are*

$$T \geq C_0 \max\{h, h^4 p^{-1/2}, h^6 p^{-1}\}$$

*different triples  $(U, V, W)$  with*

$$|U| \leq 3h, \quad |V| \leq 3h^2, \quad |W| \leq h^3,$$

*such that*

$$Us^2 + Vs + W \equiv 0 \pmod{p},$$

*then there are integers  $r$  and  $t$  satisfying conditions*

$$0 < |r| \ll h^{3/2} T^{-1/2}, \quad |t| \ll h^{5/2} T^{-1/2}, \quad rs \equiv t \pmod{p}.$$

*Proof.* We remark that if we fix  $U$  and  $V$ , then there are at most  $2h^3/p + 1$  possible values for  $W$ . In particular, we have

$$T < 36h^3(2h^3p^{-1} + 1).$$

Thus, since  $C_0$  is sufficiently large, we see that  $h < c_0p^{1/3}$  for some small positive constant  $c_0$ .

We define the lattice

$$\Gamma = \{(u, v, w) \in \mathbb{Z}^3 : us^2 + vs + w \equiv 0 \pmod{p}\}$$

and the body

$$D = \{(u, v, w) \in \mathbb{Z}^3 : |u| \leq 3h, |v| \leq 3h^2, |w| \leq h^3\}.$$

We know that

$$\#(D \cap \Gamma) \geq T.$$

Therefore, by Lemma 1, the successive minima  $\lambda_i = \lambda_i(D, \Gamma)$ ,  $i = 1, 2, 3$ , satisfy the inequality

$$(13) \quad \prod_{i=1}^n \min\{1, \lambda_i\} \ll T^{-1}.$$

We can assume that  $T$  is sufficiently large. In particular,  $\lambda_1 \leq 1$ . We consider separately the following four cases.

*Case 1:*  $\lambda_2 > 1$ . Then the inequality (13) tells us that  $\lambda_1 \ll T^{-1}$ . By definition of  $\lambda_1$ , there is a nonzero vector  $(u, v, w) \in \lambda_1 D \cap \Gamma$ . We have

$$|u| \ll hT^{-1}, \quad |v| \ll h^2T^{-1}, \quad |w| \ll h^3T^{-1}.$$

Thus, assuming that  $C_0$  is large enough, we see that  $u = 0$ . Since  $T \gg h$ , we see that  $r = -v$ ,  $t = w$  satisfy the desired bound.

*Case 2:*  $\lambda_1 < 1/(3h)$ ,  $\lambda_2 \leq 1$ , and  $\lambda_3 > 1$ . By definition of  $\lambda_1$  and  $\lambda_2$ , there are linearly independent vectors  $(u_1, v_1, w_1) \in \lambda_1 D \cap \Gamma$  and  $(u_2, v_2, w_2) \in \lambda_2 D \cap \Gamma$ . Moreover,  $\gcd(u_1, v_1, w_1) = 1$ . We see that  $|u_1| \leq \lambda_1(3h) < 1$ . Hence,  $u_1 = 0$ . We observe that  $\lambda_2 \geq 1/(3h)$ . Indeed assume that this is not true. Then  $u_2 = 0$  and we get that

$$v_i s + w_i \equiv 0 \pmod{p}, \quad i = 1, 2.$$

Hence,  $v_1 w_2 \equiv v_2 w_1 \pmod{p}$ . Since the absolute values of the both hand side is not greater than

$$3\lambda_1 \lambda_2 h^5 < h^3/3 < p/3,$$

we obtain that  $v_1 w_2 = v_2 w_1$ . This contradicts the fact that  $(u_1, v_1, w_1)$  and  $(u_2, v_2, w_2)$  are linearly independent. We also note that  $\lambda_1 \geq$

$1/(3h^2)$ , since otherwise  $u_1 = v_1 = 0$ , implying  $w_1 = 0$  (as  $w_1 \equiv 0 \pmod{p}$ ) and  $|w_1| \leq h^3 < p$ ). In particular,

$$(14) \quad \lambda_1 \lambda_2 > 1/(9h^3).$$

By (13), we have

$$(15) \quad \lambda_1 \lambda_2 \ll T^{-1}.$$

We consider the polynomials

$$P_i(Z) = u_i Z^2 + v_i Z + w_i, \quad i = 1, 2.$$

We see that  $\deg P_1 = 1$  and  $1 \leq \deg P_2 \leq 2$ . If  $\deg P_2 = 1$  then we conclude from Lemma 2 that

$$|\text{Res}(P_1, P_2)| \ll h^5 \lambda_1 \lambda_2.$$

Using (14) we get

$$|\text{Res}(P_1, P_2)| \ll h^8 (\lambda_1 \lambda_2)^2.$$

By (15),

$$(16) \quad |\text{Res}(P_1, P_2)| \ll h^8 T^{-2}.$$

If  $\deg P_2 = 2$  then we conclude from Lemma 2 that

$$|\text{Res}(P_1, P_2)| \ll h^7 \lambda_1^2 \lambda_2 \leq h^7 (\lambda_1 \lambda_2)^2,$$

and due to (15), we get (16) again. On the other hand, we see that  $\text{Res}(P_1, P_2)$  is divisible by  $p$  since  $P_1(s) \equiv P_2(s) \equiv 0 \pmod{p}$ . If  $C_0$  is chosen to be large enough, we conclude that

$$\text{Res}(P_1, P_2) = 0.$$

Therefore,  $\deg P_2 = 2$  and  $P_2$  is divisible by  $P_1$  in  $\mathbb{Z}[Z]$ . Thus,

$$u_2 w_1^2 - v_2 v_1 w_1 + w_2 v_1^2 = 0.$$

Hence, in view of  $\gcd(v_1, w_1) = 1$ , we get, for some integer  $k \neq 0$ ,

$$(17) \quad u_2 = v_1 k, \quad k w_1^2 - v_2 w_1 + w_2 v_1 = 0.$$

We recall that

$$|u_2| \ll \lambda_2 h, \quad |v_2| \ll \lambda_2 h^2, \quad |w_2| \ll \lambda_2 h^3.$$

In particular, from the first equality of (17) we get

$$|v_1| \leq |u_2| \ll \lambda_2 h.$$

Together with  $|v_1| \ll \lambda_1 h^2$ , we get that

$$|v_1| \leq (\lambda_1 \lambda_2 h^3)^{1/2}.$$

Now, the second equality of (17) implies that

$$|w_1| \leq 2|v_2| + 2|w_2 v_1|^{1/2} \ll \lambda_2 h^2 + (\lambda_1 \lambda_2 h^5)^{1/2}.$$

Combining this with  $|w_1| \ll \lambda_1 h^3$  we obtain that

$$|w_1| \ll (\lambda_1 \lambda_2 h^5)^{1/2}.$$

Consequently, by (15)

$$|v_1| \ll (\lambda_1 \lambda_2 h^3)^{1/2} \ll h^{3/2} T^{-1/2}, \quad |w_1| \ll (\lambda_1 \lambda_2 h^5)^{1/2} \ll h^{5/2} T^{-1/2},$$

and we obtain the required inequality for  $r = -v_1$ ,  $t = w_1$ .

*Case 3:*  $\lambda_1 \geq 1/(3h)$ ,  $\lambda_2 \leq 1$ , and  $\lambda_3 > 1$ . Note that (15) still holds. Hence,  $\lambda_1 \leq \lambda_2 \ll hT^{-1}$ . By definition of  $\lambda_1$  and  $\lambda_2$ , there are linearly independent vectors  $(u_1, v_1, w_1) \in \lambda_1 D \cap \Gamma$  and  $(u_2, v_2, w_2) \in \lambda_2 D \cap \Gamma$ . We have

$$|u_i| \ll h^2 T^{-1}, \quad |v_i| \ll h^3 T^{-1}, \quad |w_i| \ll h^4 T^{-1}, \quad i = 1, 2.$$

As in *Case 2*, we consider polynomials

$$P_i(Z) = u_i Z^2 + v_i Z + w_i, \quad i = 1, 2,$$

and prove that  $\text{Res}(P_1, P_2) = 0$  and thus  $P_1$  and  $P_2$  have the same linear factor  $rZ + t$  with  $\gcd(r, t) = 1$ . In particular,

$$rs + t \equiv 0 \pmod{p}.$$

Next, for some  $i \in \{1, 2\}$  we have  $u_i \neq 0$ . For this  $i$ , the equality

$$u_i t^2 - v_i t r + w_i r^2 = 0,$$

implies that

$$u_i = r k,$$

for some integer  $k \neq 0$ . In particular

$$|r| \leq |u_i| \leq \lambda_i h \ll h^2/T, \quad |k| \leq |u_i| \leq \lambda_i h.$$

Furthermore,

$$k t^2 - v_i t + w_i r = 0,$$

implying

$$|t| \leq 2|v_i| + 2(|w_i r|)^{1/2} \ll \lambda_i h^2 + (\lambda_i h^3 \lambda_i h)^{1/2} \ll \lambda_i h^2 \ll h^3/T.$$

This produces the required  $r$  and  $t$  with

$$|r| \ll h^2 T^{-1} \quad \text{and} \quad |t| \ll h^3 T^{-1}.$$

that satisfy the desired bound (since  $T \gg h$ ).

*Case 4:*  $\lambda_3 \leq 1$ . By definition of  $\lambda_i$ , there are linearly independent vectors  $(u_i, v_i, w_i) \in \lambda_i D \cap \Gamma$ ,  $i = 1, 2, 3$ . By (13), we have  $\lambda_1 \lambda_2 \lambda_3 \ll T^{-1}$ . We consider the determinant

$$D = \det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}.$$

Clearly,

$$|D| \ll h^6 \lambda_1 \lambda_2 \lambda_3 \ll h^6 T^{-1}.$$

On the other hand, from

$$u_i s^2 + v_i s + w_i = 0, \quad i = 1, 2, 3,$$

we conclude that  $D$  is divisible by  $p$ . Therefore, for a sufficiently large  $C_0$  we derive  $D = 0$ , but this contradicts linear independence of the vectors  $(u_i, v_i, w_i)$ ,  $i = 1, 2, 3$ . Thus this case is impossible.  $\square$

**3.4. Products with variables from intervals.** First consider the case of rational values of  $\sigma$  and obtain an upper bound for the number of solutions of the equation (3) satisfying (10). Then we consider the case of irrational  $\sigma$  and show that in this case that number is essentially smaller than the number of trivial solutions of (3).

**Lemma 22.** *Let  $\nu \geq 1$  be a fixed integer. Assume that  $\sigma = t/r$  for some integers  $r \geq 1$  and  $t$  with  $\gcd(r, t) = 1$ . Given an interval  $\mathcal{I} = \{x_0 + 1, \dots, x_0 + u\}$ , we fix some  $v \in \mathcal{I}$ . Then the number  $I$  of solutions of the equation*

$$(x_1 + \sigma) \dots (x_\nu + \sigma) = (y_1 + \sigma) \dots (y_\nu + \sigma) \neq 0,$$

with

$$x_1 = v, \quad x_2, \dots, x_\nu, y_1, \dots, y_\nu \in \mathcal{I},$$

and satisfying (10), does not exceed

$$I \leq \frac{u^\nu}{|vr + t|} \exp \left( O \left( \frac{\log(u + 2)}{\log \log(u + 2)} \right) \right)$$

*Proof.* We rewrite the above equation as

$$(18) \quad (x_1 r + t) \dots (x_\nu r + t) = (y_1 r + t) \dots (y_\nu r + t).$$

Given a solution to the equation (18), we fix some  $j = 1, \dots, \nu$  and for  $x_j$  consider  $X_j = |x_j r + t|$  (note that  $X_j \neq 0$ ). Taking into account that for  $i = 1, \dots, \nu$

$$y_i r + t \equiv (y_i - x_j) r \pmod{X_j},$$

we conclude from (18) that  $(y_1 - x_j) \dots (y_\nu - x_j) r^\nu \equiv 0 \pmod{X_j}$ . Clearly  $\gcd(r, X_j) = 1$ . Therefore,  $(y_1 - x_j) \dots (y_\nu - x_j) \equiv 0 \pmod{X_j}$ . In particular,  $(y_1 - v) \dots (y_\nu - v) \equiv 0 \pmod{X_j}$ .

This implies that

$$(19) \quad X_j = |x_j r + t| \leq u^\nu, \quad j = 1, \dots, \nu.$$

We now write  $(y_1 - v) \dots (y_\nu - v) = X_1 w$  for some nonzero integer  $w$  with  $|w| < h^\nu / X_1$ . Therefore, by the well-known bound on the divisor function, for each fixed  $w$  we have at most  $\exp(O(\log h / \log \log h))$

possibilities for  $y_1, \dots, y_\nu$ . Thus the total number of possibilities for  $y_1, \dots, y_\nu$  at most

$$\frac{h^\nu}{X_1} \exp(O(\log h / \log \log h)) = \frac{h^\nu}{|vr + t|} \exp(O(\log h / \log \log h)).$$

When  $y_1, \dots, y_\nu$  are fixed, using the bound (19) and the bound on the divisor function we obtain  $\exp(O(\log h / \log \log h))$  possibilities for  $x_1, \dots, x_\nu$ , which concludes the proof.  $\square$

**Lemma 23.** *Let  $\nu \geq 1$  be a fixed integer. Assume that  $\sigma = t/r$  for some integers  $r \geq 1$  and  $t$  with  $\gcd(r, t) = 1$ . Then, for  $h \geq 3$ , the number  $N$  of solutions of the equation (3) satisfying (10) does not exceed*

$$N \leq \frac{h^{\nu+1}}{\max\{hr, |t|\}} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right).$$

*Proof.* We take a reordering  $\{z_1, \dots, z_h\}$  of the elements from  $\{1, \dots, h\}$  so that

$$(20) \quad |z_1 r + t| \leq \dots \leq |z_h r + t|.$$

We notice that for any  $u = 1, \dots, h$  the set  $\{z_1, \dots, z_u\}$  is a set of  $u$  consecutive integers. For  $u = 1, \dots, h$  we denote by  $N_u$  the number of solutions of (3) satisfying (10) such that  $\{x_1, \dots, x_\nu, y_1, \dots, y_\nu\} \subseteq \{z_1, \dots, z_u\}$  and either  $x_i = z_u$  or  $y_i = z_u$  for some  $i = 1, \dots, u$ . Clearly,

$$(21) \quad N_1 = 0, \quad N = \sum_{u=2}^h N_u, \quad N_u \leq 2\nu N_u^* (u \geq 1)$$

where  $N_u^*$  is the number of solutions of (18) satisfying (10) with  $x_1 = z_u$  and  $\mathcal{I} = \{z_1, \dots, z_u\}$ . By Lemma 22, we have

$$N_u \leq \frac{u^\nu}{|z_u r + t|} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right).$$

If  $|t| \geq 2hr$  then  $|z_u r + t| \geq |t/2|$  for any  $u$ , and we get from (21)

$$N \leq \sum_{u=2}^h \frac{2u^\nu}{|t|} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right) \leq \frac{2h^{\nu+1}}{|t|} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right).$$

If  $|t| < 2hr$  then, using that there cannot be three consecutive equal elements in the sequence (20) we obtain the inequality

$$|z_u r + t| \geq (u-1)r/2 \geq ur/4$$

which yields

$$N \leq \sum_{u=2}^h \frac{4u^{\nu-1}}{r} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right) \leq \frac{4h^\nu}{r} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)$$



and completes the proof.  $\square$

Next, we need a bound on the number of solutions  $N(h)$  to the equations

$$uv = xy, \quad 1 \leq u, v, x, y \leq h,$$

which is given in [1, Theorem 3]:

**Lemma 24.** *For  $h > 1$ , we have*

$$N(h) = \frac{12}{\pi^2} h^2 \log h + \kappa h^2 + O(h^{19/13} (\log h)^{19/13}).$$

for some constant  $\kappa$ .

Note that [1, Theorem 3] gives an explicit value of  $\kappa$ . Furthermore, the error term in the asymptotic formula of Lemma 24 has recently been improved in [17], but this has no implication on our results.

Now we consider irrational values of  $\sigma$ .

We say that a solution of the equation (3) is trivial if  $(y_1, \dots, y_\nu)$  is a permutation of  $(x_1, \dots, x_\nu)$  and nontrivial otherwise. It is easy to see that in the case when  $\sigma$  is transcendental or algebraic of degree  $d \geq \nu$ , the equation (3) has only trivial solutions. Thus, it is enough to consider the case when  $\sigma$  is of degree  $d < \nu$ .

**Lemma 25.** *For every  $\nu \geq d \geq 2$ ,  $h \geq 3$  and algebraic number  $\sigma$  of degree  $d$ , the number  $N_\nu(h, \sigma)$  of solutions of the equation (3) satisfying (10) does not exceed*

$$N_\nu(h, \sigma) \leq h^{\nu-d+1} \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right).$$

*Proof.* Let  $P(X) \in \mathbb{Z}[X]$  be an irreducible primitive polynomial such that  $P(\sigma) = 0$ . Clearly,  $\deg P = d$ . Next, we consider the polynomial

$$G(X) = (x_1 + X) \dots (x_\nu + X) - (y_1 + X) \dots (y_\nu + X)$$

and factor  $G$  into irreducible over  $\mathbb{Q}$  polynomials  $f \in \mathbb{Z}[X]$ . Since  $G(\sigma) = 0$ , for some irreducible factor  $f \in \mathbb{Z}[X]$  of  $G$  we have  $f(\sigma) = 0$ . Thus,  $f(X) = rP(X)$  for some rational number  $r$ , and since  $P$  is primitive,  $r$  is integer. Thus,  $G(X) = A(X)P(X)$  for some  $A \in \mathbb{Z}[X]$ .

We have

$$A(X) = \sum_{j=0}^{\nu-d-1} a_j X^{\nu-j}.$$

By Lemma 6, we have

$$(22) \quad a_j \ll h^{j+1} \quad j = 0, \dots, \nu - d - 1,$$

where the implied constants depend only on  $\nu$ .

For  $v = 1, \dots, h$ , we now estimate the number  $N_v$  of solutions of the equation (3) satisfying (10) with  $x_1 = v$ . Clearly,

$$(23) \quad N = \sum_{v=1}^h N_v.$$

From  $G(-v) = P(-v)A(-v)$  we get that

$$-(y_1 - v) \dots (y_\nu - v) = P(-v)A(-v).$$

By (22) we have  $A(-v) \ll h^{\nu-d}$ . Therefore, there are  $O(h^{\nu-d})$  possible values for  $|(y_1 - v) \dots (y_\nu - v)|$ . In turn, this implies that there are at most  $h^{\nu-d} \exp(O(\log h / \log \log h))$  possible values for  $y_1, \dots, y_\nu$ . Once the variables  $y_1, \dots, y_\nu$  are fixed, by Lemma 3 we see that there are  $\exp(O(\log h / \log \log h))$  possibilities for  $x_2, \dots, x_\nu$  (while  $x_1$  is fixed by  $x_1 = v$ ). Thus,

$$N_v \leq h^{\nu-d} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right), \quad v = 1, \dots, h,$$

and using (23) completes the proof of the lemma.  $\square$

We say that a solution  $x_1, \dots, x_\nu, y_1, \dots, y_\nu$  to the equation (3) is *trivial* if  $(y_1, \dots, y_\nu)$  is a permutation of  $(x_1, \dots, x_\nu)$ .

**Theorem 26.** *For every  $\nu \geq d \geq 2$ ,  $h \geq 3$  and algebraic number  $\sigma$  of degree  $d$  the number  $M_\nu(h, \sigma)$  of nontrivial solutions of (3) satisfies the inequality*

$$M_\nu(h, \sigma) \leq h^{\nu-d+1} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right).$$

*Proof.* We use induction on  $\nu \geq d$ . For  $\nu = d$  all solutions are trivial, and there is nothing to prove. We verify the assertion for  $\nu > d$  assuming that it holds for  $\nu - 1$ . Using induction hypothesis we conclude that the number of nontrivial solutions of (3) such that condition (10) does not hold is bounded by

$$\begin{aligned} (2h\nu) \cdot h^{\nu-1-d+1} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right) \\ = h^{\nu-d+1} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right). \end{aligned}$$

It suffices to add the number of solutions of (3) satisfying (10). Using Lemma 25 we complete the proof.  $\square$

Since the number of trivial solutions of (3) is  $\nu!h^\nu + O(h^{\nu-1})$ , we get the following:

**Corollary 27.** *For every  $\nu \geq 1$ ,  $h \geq 3$  and irrational number  $\sigma$  of degree  $d$  we have the asymptotic formula*

$$K_\nu(h, \sigma) = \nu! h^\nu + O\left(h^{\nu-1} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right),$$

where the implicit constants depend only on  $\nu$ .

**Remark 28.** *It is certainly interesting to find best possible value of the exponent of  $h$  in Theorem 26.*

**Remark 29.** *It is also interesting to understand for which  $d$  and  $\nu$  there are nontrivial solutions of (3) for every algebraic number  $\sigma$  of degree  $d$  and a sufficiently large  $h$ .*

**Remark 30.** *One can try to get more precise forms of Theorem 26 and Lemma 25 depending on the coefficients of the minimal polynomial  $P$  for  $\sigma$  (similarly to the estimates in Lemma 22). Such an improvement is of independent interest, however does not help the main goal of this work.*

#### 4. MULTIPLICATIVE CONGRUENCES AND EQUATIONS FOR ALMOST ALL PARAMETERS

##### 4.1. Bounds on the number of solutions of multiplicative congruences for almost all $p$ .

**Theorem 31.** *Let  $\nu \geq 1$  be a fixed integer. Then for a sufficiently large positive integer  $T$ ,  $h \geq 3$ , for all but  $o(T/\log^2 T)$  primes  $p \leq T$ , if  $3 \leq h < T$  then for any  $s \in \mathbb{F}_p$ , we have the bound*

$$K_\nu(p, h, s) \leq (h^\nu + h^{2\nu-1/2} T^{-1/2}) \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right).$$

*Proof.* We note that for  $\nu = 1$  the result is trivial and we prove it for  $\nu \geq 2$  by induction on  $\nu$ .

Let

$$H_\nu = T^{1/(2\nu-1)} (\log T)^{-5/(2\nu-1)}.$$

We consider the quadruples of polynomials  $(P_1, Q_1, P_2, Q_2)$  with

$$(24) \quad P_i(Z) = (x_{1,i} + Z) \cdots (x_{\nu,i} + Z), \quad Q_i(Z) = (y_{1,i} + Z) \cdots (y_{\nu,i} + Z),$$

and

$$1 \leq x_{1,i}, \dots, x_{\nu,i}, y_{1,i}, \dots, y_{\nu,i} \leq 2h,$$

for  $i = 1, 2$ . Denote

$$R_1 = P_1 - Q_1, \quad R_2 = P_2 - Q_2.$$

Clearly, we have  $|\text{Res}(R_1, R_2)| = h^{O(1)}$ . If  $h \leq H_\nu$ , then there are at most  $O(H_\nu^{2\nu-1} \log H_\nu) = o(T/\log^2 T)$  primes  $p$  that divide  $\text{Res}(R_1, R_2)$  with  $|\text{Res}(R_1, R_2)| \neq 0$  for at least  $h^{2\nu+1}$  quadruples  $(P_1, Q_1, P_2, Q_2)$  (since each quadruples may correspond to at most  $O(\log H_\nu)$  distinct primes and we have  $O(h^{4\nu})$  distinct quadruples). If  $h > H_\nu$ , then there are at most  $o(T/\log^2 T)$  primes  $p$  that divide  $\text{Res}(R_1, R_2)$  with  $|\text{Res}(R_1, R_2)| \neq 0$  for at least  $h^{4\nu} \log^4 T/T$  quadruples  $(P_1, Q_1, P_2, Q_2)$ . Also, by the induction hypothesis, there are  $o(T/\log^2 T)$  primes  $p$  not satisfying the condition

$$(25) \quad K_{\nu-1}(p, h, s) \leq (h^{\nu-1} + h^{2\nu-5/2} T^{-1/2}) \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right).$$

We now fix one of the remaining primes  $p \leq T$  and estimate, for any  $s \in \mathbb{F}_p$ , the cardinality  $N$  of the set  $\mathcal{Q}$  of quadruples of polynomials  $(P_1, Q_1, P_2, Q_2)$  such that for  $i = 1, 2$  the polynomials  $P = P_i, Q = Q_i$  are of the form (24) with  $x_{1,i}, \dots, x_{\nu,i}, y_{1,i}, \dots, y_{\nu,i}$  that satisfy (1) and (10), and, moreover,  $\text{Res}(R_1, R_2) \neq 0$ . However,  $\text{Res}(R_1, R_2) \equiv 0 \pmod{p}$ . For any such quadruple, for any  $t = 1, \dots, h$ , and for  $i = 1, 2$  we consider polynomials

$$P_{i,t}(Z) = P_i(Z+t), \quad Q_{i,t}(Z) = Q_i(Z+t), \quad R_{i,t}(Z) = R_i(Z+t).$$

We get  $Nh$  quadruples  $(P_{1,t}, Q_{1,t}, P_{2,t}, Q_{2,t})$ . They are not necessarily distinct, but the multiplicity of any quadruple is at most  $\nu$  since  $R_{i,t}(s-t) \equiv 0 \pmod{p}$  and any polynomial  $R_i$  cannot coincide with  $R_{i,t}$  for more than  $\nu$  distinct  $t$ . Thus, for  $h \leq H_\nu$  we have  $Nh/\nu < h^{2\nu+1}$ , or

$$(26) \quad N < \nu h^{2\nu}.$$

For  $h > H_\nu$  we have  $Nh/\nu < h^{4\nu} \log^4 T/T$ , or

$$(27) \quad N < \nu h^{4\nu-1} T^{-1} \log^4 T.$$

Let  $M$  be the cardinality of the set  $\mathcal{P}$  of solutions of (1) satisfying (10). We assume that  $M \geq \nu(h^\nu + h^{2\nu-1/2} T^{-1/2} \log^2 T)$ . By the Dirichlet pigeon-hole principle, there exists  $(P_1, Q_1) \in \mathcal{P}$  such that the number of pairs  $(P_2, Q_2) \in \mathcal{P}$  with  $(P_1, Q_1, P_2, Q_2) \in \mathcal{Q}$  is at most  $h^{2\nu-1/2} T^{-1/2} \log^2 T$ . Therefore, the number  $M_0$  of pairs  $(R_1, R_2) \in \mathcal{P}$  with  $\text{Res}(R_1, R_2) = 0$  satisfies  $M_0 \geq M - h^{2\nu-1/2} \log^2 T \geq M/2$ . Thus, by Corollary 5, we find an algebraic number  $\beta$  of logarithmic height  $O(\log h)$  in an extension  $\mathbb{K}$  of  $\mathbb{Q}$  of degree  $[\mathbb{K} : \mathbb{Q}] \leq \nu$  such that the equation

$$(28) \quad (x_1 + \beta) \dots (x_\nu + \beta) = (y_1 + \beta) \dots (y_\nu + \beta) \neq 0,$$

where

$$1 \leq x_i, y_i \leq h \quad i = 1, \dots, \nu,$$

has at least  $M_0/\nu$  solutions. Now we have that

$$\beta = \frac{\alpha}{q},$$

where  $\alpha$  is an algebraic integer of height at most  $O(\log h)$  and  $q$  is a positive integer  $q \ll h^\nu$ , see [15]. From the basic properties of algebraic numbers it now follows that the numbers

$$qx_i + \alpha \quad \text{and} \quad qy_i + \alpha, \quad i = 1, \dots, \nu,$$

are algebraic integers of  $\mathbb{K}$  of height at most  $O(\log h)$ .

Using Lemma 3, we conclude that for a sufficiently large  $h$  the equation (28) has at most

$$h^\nu \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right)$$

solutions. Therefore, the same estimate holds for the number of solutions of (1) satisfying (10). By (25) we have a similar estimate for the number of solutions of (1) not satisfying (10). This completes the proof of the theorem.  $\square$

Taking the sum over  $h = 3 \times 2^j$ ,  $j \geq 0$ , we get the same exceptional set for all  $h$ .

**Corollary 32.** *Let  $\nu \geq 1$  be a fixed integer. Then for a sufficiently large positive integer  $T$ , for all but  $o(\pi(T))$  primes  $p \leq T$ , for any  $3 \leq h < T$  and for any  $s \in \mathbb{F}_p$  we have the bound*

$$K_\nu(p, h, s) \leq (h^\nu + h^{2\nu-1/2}T^{-1/2}) \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right).$$

Clearly for  $h = O(T^{1/(2\nu-1)})$  the first term in Theorem 31 and Corollary 32 dominates and both bounds take an almost optimal form

$$K_\nu(p, h, s) \leq h^\nu \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right).$$

For a set  $\mathcal{A} \subseteq \mathbb{F}_p$  we denote

$$\mathcal{A}^{(\nu)} = \{a_1 \dots a_\nu : a_1, \dots, a_\nu \in \mathcal{A}\}.$$

**Corollary 33.** *Let  $\nu \geq 1$  be a fixed integer. Then for a sufficiently large positive integer  $T$ , for all but  $o(\pi(T))$  primes  $p \leq T$ , if  $3 \leq h < T$  then for any  $s \in \mathbb{F}_p$ , for the set*

$$\mathcal{A} = \{x + s : 1 \leq x \leq h\} \subseteq \mathbb{F}_p$$

*we have*

$$\#\mathcal{A}^{(\nu)} \geq \min(h^\nu, (hT)^{1/2}) \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right).$$

#### 4.2. Asymptotic formula for the number of solutions of multiplicative equations for almost all $\sigma \in \mathbb{C}$ .

**Theorem 34.** *For every  $\nu \geq 1$ ,  $h \geq 3$  and  $\varepsilon$  with  $2 \geq \varepsilon > 0$  for all but  $O(h^{1+\varepsilon})$  values of  $\sigma \in \mathbb{C}$  we have the asymptotic formula*

$$K_\nu(h, \sigma) = \nu! h^\nu + O\left(h^{\nu-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right).$$

*Proof.* By Corollary 27, we have the desired estimate for irrational  $\sigma$ , so it is enough to consider only rational  $\sigma$ .

We denote

$$\mathcal{S} = \{s = t/r : r, t \in \mathbb{Z}, |t| \leq h^{1+\varepsilon/2}, 1 \leq r \leq h^{\varepsilon/2}\}.$$

Clearly,  $\#\mathcal{S} = O(h^{1+\varepsilon})$ . It suffices to prove the desired asymptotic formula for  $s \in \mathbb{Q} \setminus \mathcal{S}$ . We follow the proof of Theorem 26. We say that the solution of (3) is *trivial* if  $(y_1, \dots, y_\nu)$  is a permutation of  $(x_1, \dots, x_\nu)$ . The number of trivial solutions is

$$\nu! h^\nu + O(h^{\nu-1}).$$

Using induction on  $\nu$ , we prove the number of nontrivial solutions is

$$O\left(h^{\nu-\varepsilon/2} \exp\left(C(\nu) \frac{\log h}{\log \log h}\right)\right),$$

where  $C(k)$  depends only on  $k$ . For  $\nu = 1$  all solutions are trivial. We prove the assertion for  $\nu > 1$  assuming that it holds for  $\nu - 1$ . Using induction hypothesis we conclude that the number of nontrivial solutions of (3) such that condition (10) does not hold is bounded by

$$O\left(h^{\nu-\varepsilon/2} \exp\left(C(\nu-1) \frac{\log h}{\log \log h}\right)\right).$$

To estimate the number  $N$  of solutions of (3) satisfying (10) we use Lemma 23. We write  $s = t/r$  where  $t \in \mathbb{Z}$ ,  $r \in \mathbb{N}$ ,  $\gcd(r, t) = 1$ . Since  $s \notin \mathcal{S}$ , we have  $\max\{hr, |t|\} > h^{1+\varepsilon/2}$ . Hence, by Lemma 23,

$$N \leq h^{\nu-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right),$$

as required.  $\square$

We now note that the bound of Theorem 34 on the size of exceptional set of  $\sigma$  is quite precise even if one restricts  $\sigma$  to integer values.

**Theorem 35.** *For every  $\nu \geq 1$  and  $1/2 > \varepsilon > 0$  for all integers  $s$  with  $1 \leq s \leq h(\log h)^{1/2-\varepsilon}$  we have*

$$K_\nu(h, s) \gg h^\nu (\log h)^{2\varepsilon}.$$

*Proof.* As before, let  $N(h)$  be the number of integer solutions of the equation

$$uv = xy, \quad 1 \leq u, v, x, y \leq h.$$

We note that

$$\begin{aligned} N(h+s) - N(s) &\leq 4\#\{(u, v, x, y) \in \mathbb{Z}^4 : uv = xy, \\ &\quad s+1 \leq u \leq s+h, 1 \leq v, x, y \leq s+h\}. \end{aligned}$$

Take an arbitrary prime  $p \geq 2(s+h)^2$ . Then, in the above range of variables, the equation  $uv = xy$  is equivalent to the congruence  $uv \equiv xy \pmod{p}$ . Using Lemma 18, we derive

$$\begin{aligned} \#\{(u, v, x, y) \in \mathbb{Z}^4 : uv = xy, s+1 \leq u \leq s+h, 1 \leq v, x, y \leq s+h\} \\ \leq K_2(h, s)^{1/4} N(h+s)^{3/4}. \end{aligned}$$

Thus, we have

$$(29) \quad N(h+s) - N(s) \leq 4K_2(h, s)^{1/4} N(h+s)^{3/4}.$$

We now consider two cases.

*Case 1:*  $s < h/2$ . Then for a sufficiently large  $h$ , by Lemma 24 have

$$N(h+s) \geq N(h) \geq h^2 \log h \quad \text{and} \quad N(s) \leq N(\lfloor h/2 \rfloor) \leq 0.5h^2 \log h.$$

Thus,

$$N(h+s) - N(s) \geq 0.5h^2 \log h.$$

By Lemma 24 again, we have

$$N(h+s) \leq 2h^2 \log h.$$

Inserting these bounds into (29), we obtain  $K_2(h, s) \gg h^2 \log h$  and the result follows in this case.

*Case 2:*  $h/2 < s \leq h \log h$ . Then from Lemma 24 we get

$$N(h+s) - N(s) \gg sh \log h.$$

By Lemma 24 we also have

$$N(h+s) \ll s^2 \log h.$$

Combining these bounds with bounds (29), we obtain  $K_2(h, s) \gg h^2(\log h)^{2\varepsilon}$ .  $\square$

**4.3. Asymptotic formula for the number of solutions of multiplicative congruences for almost all  $s \in \mathbb{F}_p$ .** We start with the cases of  $\nu = 2$  and  $\nu = 3$  where we have stronger results than in the case of arbitrary  $\nu$ .

**Theorem 36.** *For every  $\varepsilon$  with  $2 > \varepsilon > 0$  and*

$$3 \leq h \leq p^{1/(2+\varepsilon/2)}$$

*for all but  $O(h^{1+\varepsilon})$  values of  $s \in \mathbb{F}_p$  we have the asymptotic formula*

$$K_2(p, h, s) = 2h^2 + O\left(h^{2-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right).$$

*Proof.* Assume that  $s$  is such that

$$(30) \quad (x_1 + s)(x_2 + s) \equiv (y_1 + s)(y_2 + s) \not\equiv 0 \pmod{p}$$

is satisfied for at least  $h^{2-\varepsilon/2} \exp(C_1 \log h / \log \log h)$  choices of integers  $1 \leq x_1, x_2, y_1, y_2 \leq h$  with  $\{x_1, x_2\} \neq \{y_1, y_2\}$ , where  $C_1 > 0$  is a sufficiently large constant.

Using Lemma 19, we see that  $s$  satisfies at least  $C_2 h^{1-\varepsilon/2}$  distinct linear congruences  $As + B \equiv 0 \pmod{p}$  with  $0 < |A| \leq 2h$  and  $|B| \leq h^2$  where  $C_2 > 0$  is another sufficiently large constant. Thus by Lemma 20, applied with  $\gamma = 16h^{-\varepsilon/2}$  we obtain that  $s$  can take at most  $O(h^{1+\varepsilon})$  possible values.  $\square$

**Theorem 37.** *For every  $\varepsilon$  with  $2 > \varepsilon > 0$  and*

$$3 \leq h \leq p^{1/(4+\varepsilon)}$$

*for all but  $O(h^{1+\varepsilon})$  values of  $s \in \mathbb{F}_p$  we have the asymptotic formula*

$$K_3(p, h, s) = 6h^3 + O\left(h^{3-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right).$$

*Proof.* By Theorem 36, we see that there is a set  $\mathcal{S} \subseteq \mathbb{F}_p$  of cardinality  $O(h^{1+\varepsilon})$  such that for all  $s \in \mathbb{F}_p \setminus \mathcal{S}$ , the equation (30) has at most  $h^{2-\varepsilon/2} \exp(O(\log h / \log \log h))$  solutions with  $\{x_1, x_2\} \neq \{y_1, y_2\}$ .

It is now easy to see that for  $s \notin \mathcal{S}$  we have

$$K_3(p, h, s) = 6h^3 + O\left(h^{3-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right) + N\right),$$

where  $N$  denotes the contribution to  $K_3(p, h, s)$  of solutions with  $x_i \neq y_j$ ,  $1 \leq i, j \leq 3$ .

Assume that

$$N \geq h^{3-\varepsilon/2} \exp\left(C_1 \frac{\log h}{\log \log h}\right)$$

for some appropriate constant  $C_1 > 0$ .



We see from Lemma 19 that for sufficiently large  $h$  and for another appropriate constant  $C_2 > 0$  there are at least  $T = \lfloor C_2 h^{2-\varepsilon/2} \rfloor$  different triples  $(U, V, W)$  with

$$|U| \leq 3h, \quad |V| \leq 3h^2, \quad |W| \leq h^3,$$

such that

$$Us^2 + Vs + W \equiv 0 \pmod{p}.$$

Clearly,  $T$  satisfies the conditions of Lemma 21, thus there are some integers  $r$  and  $t$  with  $0 < |r| \ll h^{1/2+\varepsilon/4}$ ,  $t \ll h^{3/2+\varepsilon/4}$ , and  $rs \equiv t \pmod{p}$ .

Clearly we can assume that  $\gcd(r, t) = 1$ .

We see that

$$(x_1 + s)(x_2 + s)(x_3 + s) \equiv (y_1 + s)(y_2 + s)(y_3 + s) \not\equiv 0 \pmod{p}$$

implies

$$(31) \quad \begin{aligned} & (x_1 + x_2 + x_3 - y_1 - y_2 - y_3)t^2 \\ & + (x_1x_2 + x_1x_3 + x_2x_3 - y_1y_2 - y_1y_3 - y_2y_3)rt \\ & + (x_1x_2x_3 - y_1y_2y_3)r^2 \equiv 0 \pmod{p}. \end{aligned}$$

The absolute value of the left-hand side of (31) is at most  $14h^{4+\varepsilon/2} < p$ , and we get the equation

$$\begin{aligned} & (x_1 + x_2 + x_3 - y_1 - y_2 - y_3)t^2 \\ & + (x_1x_2 + x_1x_3 + x_2x_3 - y_1y_2 - y_1y_3 - y_2y_3)rt \\ & + (x_1x_2x_3 - y_1y_2y_3)r^2 = 0, \end{aligned}$$

which, in turn, we transform into an equivalent equation

$$(x_1 + \sigma)(x_2 + \sigma)(x_3 + \sigma) = (y_1 + \sigma)(y_2 + \sigma)(y_3 + \sigma)$$

with a rational  $\sigma = t/r$ . Since  $\gcd(t, r) = 1$  different values of  $s$  lead to different values of  $\sigma$ . Using Theorem 34 we conclude the proof.  $\square$

Furthermore, we have the following general form of Theorem 37 which holds for an arbitrary  $\nu \geq 1$ .

**Theorem 38.** *For every  $\nu \geq 2$ , there exists some positive  $\gamma(\nu)$  such that for*

$$3 \leq h \leq \gamma(\nu)p^{1/(\nu^2-1)}$$

*and  $0 < \varepsilon \leq 2$  for all but  $O(h^{1+\varepsilon})$  values of  $s \in \mathbb{F}_p$  we have the asymptotic formula*

$$K_\nu(p, h, s) = \nu!h^\nu + O\left(h^{\nu-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right).$$

*Proof.* Without loss of generality we may assume that  $p$  is large enough. As before, we say that a solution of (2) is *trivial* if  $(y_1, \dots, y_\nu)$  is a permutation of  $(x_1, \dots, x_\nu)$ . Let  $S$  be the set of all  $s \in \mathbb{F}_p$  such that (2) has at least one nontrivial solution. If  $s \in \mathbb{F}_p \setminus \{S\}$  then

$$K_\nu(p, h, s) = \nu!h^\nu + O(h^{\nu-1}).$$

Take  $s \in S$  and fix a nontrivial solution  $(x_1^*, \dots, x_\nu^*, y_1^*, \dots, y_\nu^*)$  of (2). Define the polynomial

$$R^*(Z) = (x_1^* + Z) \dots (x_\nu^* + Z) - (y_1^* + Z) \dots (y_\nu^* + Z).$$

Clearly,  $R^*$  is not a zero polynomial and  $R^*(s) \equiv 0 \pmod{p}$ . In particular, since  $1/(\nu^2 - 1) < 1/\nu$  for  $\nu \geq 2$  we see that  $R^*$  is not a constant polynomial, assuming that  $p$  is sufficiently large.

We decompose  $R^*$  as a product of irreducible over  $\mathbb{Q}$  polynomials

$$R^*(z) = W_1(Z) \dots W_n(Z)$$

with  $W_j \in \mathbb{Z}[Z]$  for  $j = 1, \dots, n$ . We have  $W_j(s) \equiv 0 \pmod{p}$  for some  $j$ . Denote  $W^* = W_j$ .

Now we consider any nontrivial solution  $(x_1, \dots, x_\nu, y_1, \dots, y_\nu)$  of (2) and define the polynomial

$$R(Z) = (x_1 + Z) \dots (x_\nu + Z) - (y_1 + Z) \dots (y_\nu + Z).$$

Again,  $R$  is not a zero polynomial and  $R(s) \equiv 0 \pmod{p}$ . As in the above we see that  $R$  is not a constant polynomial.

Writing

$$R(Z) = \sum_{j=0}^{\nu-1} r_j Z^{\nu-1-j}$$

we see that

$$(32) \quad r_j \ll h^{j+1}, \quad j = 0, \dots, \nu - 1.$$

Clearly  $R^*$  satisfies the same bound. So writing

$$W^*(Z) = \sum_{j=0}^{\mu-1} w_j Z^{\mu-1-j}$$

for some  $\mu \leq \nu$  and applying Lemma 6, we infer

$$(33) \quad w_j \ll h^{j+1}, \quad j = 0, \dots, \mu - 1.$$

We have  $\text{Res}(R, W^*) \equiv 0 \pmod{p}$  since  $R(s) \equiv W^*(s) \equiv 0 \pmod{p}$ . Next, by Lemma 2, (applied with  $\rho = \vartheta = 1$ ) we see that

$$\text{Res}(R, W^*) \ll h^{\nu^2-1}.$$

Thus, provided that  $\gamma(\nu)$  is small enough and  $p$  is large enough we obtain the inequality

$$|\text{Res}(R, W^*)| < p.$$

Hence,  $\text{Res}(R, W^*) = 0$ . Using irreducibility of the polynomial  $W^*$  we conclude that  $R$  is divisible by  $W^*$  in  $\mathbb{Q}[Z]$ .

Consider a mapping  $\Phi : S \rightarrow \mathbb{C}$  by associating with any  $s \in S$  a zero  $\sigma$  of a corresponding polynomial  $W^*$ . We see from the above discussion that any solution  $(x_1, \dots, x_\nu, y_1, \dots, y_\nu)$  of congruence (2) induces the same solution of the equation (3). Thus,

$$K_\nu(p, h, s) = K_\nu(h, \sigma).$$

Using Theorem 34, we get

$$K_\nu(p, h, s) = \nu!h^\nu + O\left(h^{\nu-\varepsilon/2} \exp\left(O\left(\frac{\log h}{\log \log h}\right)\right)\right)$$

unless  $\sigma = \Phi(s)$  is an element of an exceptional subset of  $\mathbb{C}$  of cardinality  $O(h^{1+\varepsilon})$ . Taking into account that a preimage  $\Phi^{-1}(\sigma)$  contains at most  $n \leq \nu - 1$  elements for any  $\sigma \in \mathbb{C}$ , we complete the proof.  $\square$

## 5. DISTRIBUTION OF ELEMENTS OF LARGE MULTIPLICATIVE ORDER

**5.1. Distribution in very short intervals for almost all  $p$ .** Let  $\text{ord}_p a$  denote the multiplicative order of  $a \in \mathbb{F}_p^*$ . Our aim is to prove that for almost all primes  $p$  very short intervals in  $\mathbb{F}_p$  (including intervals of fixed size) contain an element of large multiplicative order.

For  $h = 2$ , Chang [7] has shown that for any function  $\eta(z) > 0$  with  $\eta(z) \rightarrow 0$  as  $z \rightarrow \infty$  and sufficiently large positive integer  $T$ , for all but  $o(\pi(T))$  primes  $p \leq T$ , for all but  $O(1)$  elements  $s \in \mathbb{F}_p$ , we have

$$(34) \quad \max\{\text{ord}_p s, \text{ord}_p(s+1)\} > p^{1/4+\eta(p)}.$$

In fact her argument implies that (34) hold for almost all primes  $p$  and for any  $s \in \mathbb{F}_p^*$  with  $\text{ord}_p s > 3$ .

We note that if  $\text{ord}_p s = 3$  then  $s+1 \equiv -s^2 \pmod{p}$  thus  $\text{ord}_p(s+1) = 6$ .

For small  $h$  we have the following result.

**Theorem 39.** *Let  $\eta(z) > 0$ ,  $m(z) > 0$  be arbitrary functions with  $\eta(z) \rightarrow 0$  and  $m(z) \rightarrow \infty$  as  $z \rightarrow \infty$ . Then for all but  $o(\pi(T))$  primes  $p \leq T$ , any interval  $I = [t, t+m(T)]$  has an element  $\xi$  for which*

$$\text{ord}_p \xi > T^{1/2+\eta(T)}.$$

*Proof.* We assume that  $T$  is large enough. Using Lemma 17 we take  $c = c(1/2) > 0$ . Note that either by increasing  $\eta(T)$  or decreasing  $m(T)$ , we may assume that  $\eta(z) > (\log z)^{-1/4}$  and that

$$(35) \quad m(T) = \lfloor \eta(T)^{-C/\eta(T)} \rfloor, \quad C = 3/c.$$

holds. Next, denote

$$(36) \quad m = m(T), \quad r = \lfloor c \log m \rfloor, \quad \mu = \lfloor m^{-1} T^{1/2(r+2)} \rfloor.$$

Given  $\mathcal{E}_1, \mathcal{E}_2 \subseteq \{1, \dots, m\}$ ,  $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$  and  $\#\mathcal{E}_1 + \#\mathcal{E}_2 \leq r$ ,  $\tilde{\mu} = (\mu_j)_{j \in \mathcal{E}_1 \cup \mathcal{E}_2}$  with  $1 \leq \mu_j \leq \mu$ , denote

$$P_{\mathcal{E}_1, \mathcal{E}_2, \tilde{\mu}}(X) = \prod_{j \in \mathcal{E}_1} (X + j)^{\mu_j} - \prod_{j \in \mathcal{E}_2} (X + j)^{\mu_j} \in \mathbb{Z}[X].$$

These polynomials are of degree at most  $r\mu$  and logarithmic height  $O(r\mu \log m)$ . Their number is clearly bounded by  $2^{r+1} \binom{m}{r} \mu^r$  (here we have used that  $r \leq m/2$ ).

Factor each polynomial  $P_{\mathcal{E}_1, \mathcal{E}_2, \tilde{\mu}}(X)$  in irreducible (over  $\mathbb{Q}$ ) factors  $f \in \mathbb{Z}[X]$  and let  $\mathfrak{P} \subseteq \mathbb{Z}[X]$  be the set of all polynomials obtained this way. Hence,

$$(37) \quad \#\mathfrak{P} \leq r 2^{r+1} \binom{m}{r} \mu^{r+1}.$$

Denote then

$$R = \prod_{\substack{f, g \in \mathfrak{P} \\ \text{Res}(f, g) \neq 0}} \text{Res}(f, g) \in \mathbb{Z}.$$

Using Corollary 5 we see that all coefficients of any polynomial  $f \in \mathfrak{P}$  are bounded by  $(O(m))^{r\mu}$ . Next, using a straightforward bound on the resultants  $\text{Res}(f, g)$ , by (36) and (37), we derive

$$|R| \leq m^{O(r^4 4^r \binom{m}{r}^2 \mu^{2(r+2)})} < 2^{o(T)}.$$

Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be the sets of exceptional primes in Lemma 10 and Corollary 12 respectively. Hence, denoting

$$\mathcal{P}_0 = \{p < T : \gcd(p, R) = 1, p \notin \mathcal{P}_1 \cup \mathcal{P}_2\},$$

we have

$$\#\mathcal{P}_0 = (1 + o(1))\pi(T).$$

Take  $p \in \mathcal{P}_0$ . Since  $p \notin \mathcal{P}_1$ , the desired statement is equivalent to

$$(38) \quad \text{ord}_p \xi \geq T^{1/2 - \eta(T)}.$$

Let  $I = [t, t + m(T)] \subseteq \mathbb{F}_p$  and assume that

$$(39) \quad \text{ord}_p \xi < T^{1/2 - \eta(T)} \quad \text{for all } \xi \in I.$$

Since  $p \notin \mathcal{P}_2$ , we may write  $p - 1 = AB$ , where  $A \leq T^{\eta(T)/2}$  and  $B$  has no prime factors less than  $T^{c\eta(T)/(\log(1/\eta(T)))}$ . Hence,  $B$  has at most  $(1/\eta(T))^{1/(c\eta(T))}$  divisors. For  $\xi \in I$  we write

$$\frac{p-1}{\text{ord}_p \xi} = \frac{AB}{\text{ord}_p \xi} = ab$$

where  $a \mid A$ ,  $b \mid B$ . In particular,  $\xi = g^{ab}$  for some primitive root  $g$  modulo  $p$ . Thus,  $\xi$  belongs to the subgroup  $\mathcal{H}$  of  $\mathbb{F}_p^*$  generated by the element  $g^b$ .

Since  $A \leq T^{\eta(T)/2}$ , using (39), we derive that

$$b = \frac{p-1}{a \text{ord}_p \xi} > \frac{p-1}{T^{(1-\eta(T))/2}}.$$

Thus,

$$\#\mathcal{H} \leq \frac{p-1}{b} < T^{(1-\eta(T))/2}.$$

Since  $B$  has at most  $(1/\eta(T))^{1/(c\eta(T))}$  divisors, the above argument shows that we have a family of at most  $(1/\eta(T))^{1/(c\eta(T))}$  subgroups  $\mathcal{H}$  of  $\mathbb{F}_p^*$  of size

$$\#\mathcal{H} < T^{(1-\eta(T))/2}$$

and such that each element  $\xi \in I$  is contained in one of these groups. Consequently, we conclude that there is a subgroup  $\mathcal{H}$  of  $\mathbb{F}_p^*$  of order

$$(40) \quad \#\mathcal{H} = N < T^{(1-\eta(T))/2}$$

such that for the set  $\mathcal{S} \subseteq [1, m]$ , defined by

$$t + \mathcal{S} = \mathcal{H} \cap I,$$

by the choice of parameters  $c$ ,  $m$  and  $r$  in (35) and (36), we have

$$\#\mathcal{S} > \sqrt{m}.$$

Let  $\mathcal{E} \subseteq \mathcal{S}$ ,  $\#\mathcal{E} = r$ . Assuming all elements

$$\prod_{j \in \mathcal{E}} (t + j)^{\mu_j} \in \mathcal{H}$$

with  $0 \leq \mu_j \leq \mu$  are distinct modulo  $p$ , it follows from (35) and (36) that

$$N \geq \mu^r > T^{(1-\eta(T))/2}$$

contradicting (40).

Hence, for each  $\mathcal{E} \subseteq \mathcal{S}$  with  $\#\mathcal{E} = r$ , there are disjoint  $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}$  and exponents  $\tilde{\mu} = (\mu_j)_{j \in \mathcal{E}_1 \cup \mathcal{E}_2}$  such that

$$P_{\mathcal{E}_1, \mathcal{E}_2, \tilde{\mu}}(t) \equiv 0 \pmod{p}.$$

We may then extract an irreducible (over  $\mathbb{Q}$ ) factor  $f_E \in \mathfrak{P}$  from  $P_{\mathcal{E}_1, \mathcal{E}_2, \tilde{\mu}}$  such that

$$f_E(t) \equiv 0 \pmod{p}.$$

Thus, for all  $\mathcal{E}, \mathcal{F} \subseteq S$  with  $\#\mathcal{E} = \#\mathcal{F} = r$

$$\text{Res}(f_{\mathcal{E}}, f_{\mathcal{F}}) \equiv 0 \pmod{p},$$

while also  $\text{Res}(f_{\mathcal{E}}, f_{\mathcal{F}}) \mid R$ . Since  $\gcd(p, R) = 1$ , it follows that necessarily

$$\text{Res}(f_{\mathcal{E}}, f_{\mathcal{F}}) = 0$$

for all  $\mathcal{E}, \mathcal{F} \subseteq S$  of size  $\#\mathcal{E} = \#\mathcal{F} = r$ . Since the  $f_{\mathcal{E}}$  are irreducible, they must coincide up to a scalar factor and hence have a common root  $\beta \in \mathbb{C}$ .

Apply then Lemma 17 to  $\mathcal{A} = S + \beta \subseteq \{\beta + j; 1 \leq j \leq m\} \subseteq \mathbb{C}$ . This gives a multiplicatively independent set  $\mathcal{A}_0 = \mathcal{E} + \beta$  with some  $\mathcal{E} \subseteq S$  of size  $\#\mathcal{E} = r$ . But since  $f_{\mathcal{E}}(\beta) = 0$ , we get  $P_{\mathcal{E}_1, \mathcal{E}_2, \tilde{\mu}}(\beta) = 0$ , contradicting the multiplicative independence.  $\square$

## 5.2. Distribution in very short intervals for a large proportion of primes $p$ .

**Theorem 40.** *If  $a \in \mathbb{N}$ ,  $m > 1$ ,  $1 \leq \mu < \log m$  and  $T \in \mathbb{Z}_+$  is taken sufficiently large, then for all but  $O(\mu^{-1}\pi(T))$  primes  $p < T$  we have*

$$\max_{0 \leq j < m} \text{ord}_p(a + j) > T^{1-m^{-1/\mu}}.$$

*Proof.* For small  $\mu$  (and, in particular, for small  $m$ ) the result is trivial. We assume that  $\mu$  is large enough. Moreover, it is enough to prove the result for

$$(41) \quad \mu < 0.1 \log m.$$

Take  $r = 4 \lfloor m^{1/\mu} \rfloor$ . Then we have

$$(42) \quad r \leq m^{0.1}.$$

Let  $\mathfrak{L}$  be the collection of all multiplicatively independent subsets  $\mathcal{S} \subseteq I = [a, a + m)$  of cardinality  $r$ . The set  $\mathfrak{L}$  is nonempty by Corollary 8. Let  $T$  be sufficiently large (depending on  $a, m$ ) and set

$$K = \lfloor T^{1/(r+2)} \rfloor,$$

$$R = \prod_{\substack{\mathcal{S} \in \mathfrak{L} \\ \mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}, \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset}} \left( \prod_{\substack{1 \leq k_{\xi} \leq K \\ \xi \in \mathcal{S}_1}} \xi^{k_{\xi}} - \prod_{\substack{1 \leq k_{\xi} \leq K \\ \xi \in \mathcal{S}_2}} \xi^{k_{\xi}} \right).$$

Hence,  $R \in \mathbb{Z} \setminus \{0\}$  and for  $T$  large enough

$$|R| < (a+m)^{rK^{r+1}3^r} < 2^{o(T)}.$$

Hence, denoting

$$\mathcal{P}_0 = \{p < T : \gcd(p, R) = 1\}$$

we have

$$(43) \quad \#\mathcal{P}_0 = (1 + o(1))\pi(T).$$

Now we take

$$\alpha = m^{-1/\mu}/\mu, \quad \beta = 1/\mu, \quad \gamma = m^{-1/\mu}/2$$

and denote by  $\mathcal{P}_1$  the set of primes  $p \in \mathcal{P}_0$  such that

- (i) the product of all prime factors of  $p-1$  that are smaller than  $T^\alpha$  is at most  $T^\gamma$ ;
- (ii) the number of prime divisors  $q$  of  $p-1$  satisfying  $q \geq T^\alpha$  does not exceed

$$\frac{0.2\mu \log(\beta/\alpha)}{1-\beta} + \frac{1}{\beta};$$

- (iii)  $p-1$  has no divisor  $q^2$  with  $q \geq T^\alpha$ .

By (43) and Lemmas 11, 13, and 14, we derive

$$\#\mathcal{P}_1 = (1 + O(1/\mu))\pi(T).$$

Also, observe that, by (41), for large enough  $\mu$  we have

$$(44) \quad \frac{0.2\mu \log(\beta/\alpha)}{1-\beta} + \frac{1}{\beta} < 0.3 \log m + \mu < 0.4 \log m.$$

By assumption (i) in the definition of  $\mathcal{P}_1$ , we may write  $p-1 = AB$ , where  $A < T^\gamma$  and  $B$  has no prime factors less than  $T^\alpha$ . By (ii) and (44),  $B$  has at most  $0.4 \log m$  prime factors. By (iii),  $B$  is square-free. Therefore, the number of factors of  $B$  is at most

$$2^{0.4 \log m} < m^{0.3}.$$

We assume that for  $j = 0, \dots, m-1$  we have

$$\text{ord}_p(a+j) \leq T^{1-m^{-1/\mu}}.$$

By Corollary 8, we choose a set

$$\mathcal{I} \subseteq \{a+j : j \in [0, m[), \quad \#\mathcal{I} \gg \sqrt{m}/\log m,$$

of multiplicatively independent numbers. For  $\xi \in \mathcal{I}$  we write

$$\frac{p-1}{\text{ord}_p \xi} = \frac{AB}{\text{ord}_p \xi} = ab$$

where  $a \mid A$ ,  $b \mid B$ . Since  $B$  has at most  $m^{0.3}$  divisors, we can take a subset  $\mathcal{J} \subseteq \mathcal{I}$  with  $\#\mathcal{J} \geq \#\mathcal{I}m^{-0.3} > m^{0.1}$  such that the same divisor  $b$  of  $B$  is associated to any  $\xi \in \mathcal{J}$ . We have  $\xi = g^{ab}$  for some primitive root  $g$  modulo  $p$ . Thus,  $\xi$  belongs to the subgroup  $\mathcal{H}$  of  $\mathbb{F}_p^*$  generated by the element  $g^b$ . We have

$$b = \frac{p-1}{a \operatorname{ord}_p \xi} > \frac{p-1}{T^\gamma \operatorname{ord}_p \xi} \geq \frac{p-1}{T^{1-0.5m^{-1/\mu}}}.$$

Thus,  $\mathcal{J} \subseteq \mathcal{H}$  and

$$\#\mathcal{H} \leq \frac{p-1}{b} < T^{1-0.5m^{-1/\mu}}.$$

By (42), we can take a subset  $\mathcal{S} \subseteq \mathcal{J}$  of cardinality  $\#\mathcal{S} = r$ . Since  $p \in \mathcal{P}_0$ , we conclude that all products

$$\prod_{\substack{0 \leq k_\xi \leq K \\ \xi \in \mathcal{S}}} \xi^{k_\xi}$$

are distinct modulo  $p$ . The number of such products is

$$(K+1)^r > T^{r/(r+2)} > T^{1-0.5m^{-1/\mu}} > \#\mathcal{H}.$$

But this is impossible since all the products belong to  $\mathcal{H}$ . This completes the proof.  $\square$

**5.3. Distribution in longer intervals for almost all  $p$ .** For large  $h$  we can use Theorem 31.

**Theorem 41.** *Let  $\alpha > 0$  be fixed. For  $T^\alpha \leq h < T$ , for all but  $o(\pi(T))$  primes  $p \leq T$  and for any  $s \in \mathbb{F}_p$ , the set*

$$\mathcal{A} = \{x + s : 1 \leq x \leq h\} \subseteq \mathbb{F}_p$$

*contains an element  $a \in \mathcal{A}$  of multiplicative order*

$$\operatorname{ord}_p a > \exp \left( O \left( \frac{\log h}{\log \log h} \right) \right) (hT)^{1/2}.$$

*Proof.* We fix  $\nu \geq 1/\alpha$ . Clearly  $\mathcal{A}$  contains a set  $\mathcal{B} \subseteq \mathcal{A}$  of elements of the same multiplicative order  $t$  and of cardinality

$$(45) \quad \#\mathcal{B} \geq \#\mathcal{A}/\tau(p-1) = h/\tau(p-1).$$

For  $\lambda \in \mathbb{F}_p^*$ , let

$$Q(\lambda) = \#\{(x_1, \dots, x_\nu) \in \mathcal{B} \times \dots \times \mathcal{B} : \lambda \equiv x_1 \dots x_\nu \pmod{p}\}.$$

Then obviously,

$$\#\{\lambda \in \mathbb{F}_p^* : Q(\lambda) > 0\} \leq t.$$



Hence, using the Cauchy inequality, we obtain

$$(\#\mathcal{B})^{2\nu} = \left( \sum_{\lambda \in \mathbb{F}_p^*} Q(\lambda) \right)^2 \leq t \sum_{\lambda \in \mathbb{F}_p^*} Q(\lambda)^2 \leq tK_\nu(p, h, s),$$

which together with Theorem 31 and the standard estimate for  $\tau(p-1)$  implies the result.  $\square$

We note that for intermediate values of  $h$ , namely for  $h$  with  $T^\alpha \leq h < T^{1-\alpha}$  for some fixed  $\alpha > 0$ , using the ideas and results of Erdős and Murty [9] one can improve slightly Theorem 41. Namely, one can show that for any function  $\eta(z) > 0$  with  $\eta(z) \rightarrow 0$  we have

$$\text{ord}_p a > (hT)^{1/2} T^{\eta(T)}$$

instead of the bound of Theorem 41.

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#### REFERENCES

- [1] A. Ayyad, T. Cochrane and Z. Zheng, ‘The congruence  $x_1x_2 \equiv x_3x_4 \pmod{p}$ , the equation  $x_1x_2 = x_3x_4$  and the mean value of character sums’, *J. Number Theory*, **59** (1996), 398–413.
- [2] U. Betke, M. Henk and J. M. Wills, ‘Successive-minima-type inequalities’, *Discr. Comput. Geom.*, **9** (1993), 165–175.
- [3] J. Bourgain, M. Z. Garaev, S. V. Konyagin and I. E. Shparlinski, ‘On the hidden shifted power problem’, *SIAM J. Comp.*, (to appear).
- [4] J. Bourgain, M. Z. Garaev, S. V. Konyagin and I. E. Shparlinski, ‘On congruences with products of variables from short intervals and applications’, *Proc. Steklov Math. Inst.*, (to appear).
- [5] M.-C. Chang, ‘Factorization in generalized arithmetic progressions and applications to the Erdős-Szemerédi sum-product problems’, *Geom. Funct. Anal.*, **13** (2003), 720–736.
- [6] M. C. Chang, ‘The Erdős-Szemerédi problem on sum set and product set’, *Ann. of Math.*, **157** (2003), 939–957.
- [7] M.-C. Chang, ‘Elements of large order in prime finite fields’, *Bull. Aust. Math. Soc.* (to appear).
- [8] J. Cilleruelo and M. Z. Garaev, ‘Concentration of points on two and three dimensional modular hyperbolas and applications’, *Geom. and Func. Anal.*, **21** (2011), 892–904.

- [9] P. Erdős and R. Murty, ‘On the order of  $a \pmod{p}$ ’, *Proc. 5th Canadian Number Theory Association Conf.*, Amer. Math. Soc., Providence, RI, 1999, 87–97.
- [10] J. H. Evertse, H. P. Schlickewei and W. M. Schmidt, ‘Linear equations in variables which lie in a multiplicative group’, *Ann. of Math.*, **155** (2002), 807–836.
- [11] J. von zur Gathen and J. Gerhard, *Modern computer algebra*, Cambridge University Press, Cambridge, 2003.
- [12] H. Iwaniec, ‘On the problem of Jacobsthal’, *Demonst. Math.*, **11** (1978), 225–231.
- [13] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc., Providence, RI, 2004.
- [14] M. Mignotte, *Mathematics for computer algebra*, Springer-Verlag, Berlin, 1992.
- [15] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Polish Sci. Publ., Warszawa, 1990.
- [16] C. Pomerance and I. E. Shparlinski, ‘Smooth orders and cryptographic applications’, *Proc. 5-th Algorithmic Number Theory Symp.*, Lect. Notes in Comput. Sci., vol. 2369, Springer-Verlag, Berlin, 2002, 338–348.
- [17] S. Shi, ‘The equation  $n_1 n_2 \equiv n_3 n_4 \pmod{p}$  and mean value of character sums’, *J. Number Theory*, **128** (2008), 313–321.
- [18] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Stud. Adv. Math., **105**, Cambridge University Press, Cambridge, 2006.

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